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METHOD OF MEASURING GEODETIC BASES BY MEANS OF METALLIC WIRES,

BY

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TRANSLATED FROM

MÉMOIRES PRÉSENTÉS PAR DIVERS SAVANTS À L'ACADÉMIE DES SCIENCES DE L'INSTITUT DE FRANCE.

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METHOD OF MEASURING

GEODETIC BASES

BY MEANS OF METALLIC WIRES

BY

M. JÄDERIN.

It is about 20 years since steel measuring tapes, of considerable length, were put on the market: these are naturally better than other measuring tapes or than surveying chains, for operations on the ground according to the ordinary method of procedure, *viz.*, allowing the chain or tape to rest immediately on the ground. Yet, in spite of the good points of the steel tape, this way of measuring will not give good results unless the surface on which the tape rests is quite flat, as for example, a sheet of ice or a line of rail. On unequal ground it would be much more difficult to carry a flat platform on supports for the tape to rest on, and even if that were done certain difficulties, in the determination of temperature for instance, would greatly diminish the value of the work.

These various inconveniences can be greatly diminished in the following manner:—First, suppose that a steel tape is stretched on a plane surface but without any tension. Its length, the rectilinear distance between its extreme graduations will be normal. Let the exact positions of the terminal points be marked on the plane surface and let these marked extremities be raised vertically so that they will become points of support, then by means of tension applied at the two ends, the tape may be kept suspended with the terminal points on these marks: it will have its normal length, that is to say, the rectilinear distance between the extreme marks will be the same as it was when the tape rested on the ground, although it now forms a curve, on account of the tension applied to it. The force employed to obtain this amount of stretching, which I call the *normal tension*, varies between 5 and 8 kilogrammes and I will mention later how I have determined it, *see* equation (23). The tension now replaces the plane surface: it can be obtained by applying spring-balances at the two ends of the tape.

Instead of working on the ground we thus measure above it in the air. At each length of tape, a tripod (fig. 1) is placed, which carries on top a brass cylinder placed vertically: the upper surface of the latter is made spherical and carries a finely cut mark in the form of a cross. The actual measurement is made between these marks placed in the line of the base at intervals equal to the length of the tape (25 metres say), the intervals being determined by a preliminary provisional measurement. One of the terminal marks on the steel tape is placed very exactly by means of a magnifying glass on the mark or index of one of the supports and on the second support the obtained distance is read by means of a millimetre scale on the tape. The distance is determined by the magnifying glass to tenths of millimetres.

In this operation, the spring-balances attached to each end of the tape ought to be handled by persons accustomed to the work who ought among other things to take care that there is no friction in the spring-balance. To obtain a sufficient stability and not to fatigue themselves during the process of measuring, they ought to attach each spring-balance to an iron-shod stake fixed in the ground and hold the upper extremity pressed behind the upper arm with the point touching the shoulder, so that properly speaking it is the weight of the body which pulls the balance. If they hold only with the hand, the pull will be variable and they will tire in a few minutes.

In passing obstacles and on arrival at the other end, when in general the whole length of the tape will not be wanted, it is clear that a new normal tension should be used.

For the reduction to the horizontal distance, the successive differences of level between the indexes of the supports and the ends of the base are determined by the ordinary process of geometrical levelling.

It is assumed that the temperature of the tape is identical with that of the surrounding air determined by means of thermometers. The corrections are made with respect to a normal temperature of + 15°C. to which the tape has been referred during comparison with the standard.

A geodetic measurement of a base made in this way gives good results in favourable circumstances, say with a cloudy sky and an absence of wind. If it blows even moderately, the tape oscillates so much that it is difficult or impossible to read the distances on the millimetre scale. If the sun is shining it may happen that the temperature of the tape is not the same as that of the air, or that the temperature of the latter cannot be determined exactly by means of the thermometer.

The effect of wind on the tape can be diminished in two ways, *firstly* by increasing the tension, taking account of the difference of length produced by this increase, *see* equation (22), *secondly* by changing the dimensions of the tape so that its breadth is diminished and its thickness is increased while the area of the cross section is maintained, thus reducing the surface which is presented to the wind.

The minimum surface is obtained in circular areas, and for this reason I have been induced to replace the steel tape by a steel wire. These steel wires, generally 25 metres long, are furnished at their two extremities with millimetre scales each a decimetre long, *see* S, (fig. 2). In certain particular cases, such as on arrival at the second end of the base, a narrow steel tape is used.

Experience proves that wires of about 1·6* millimetres stretched with a force equivalent to 10 kilogrammes are completely insensible even to strong wind, so that by employing these we get rid of the first of the above-mentioned inconveniences.

We can get over the second difficulty which arises from the temperature due to the sun by using another wire along with the steel wire which we will call A. The second wire B will be of a different metal, that is, it will possess a coefficient of expansion sensibly different from A:

* ·063 inch or about No. 16 B.W.G.

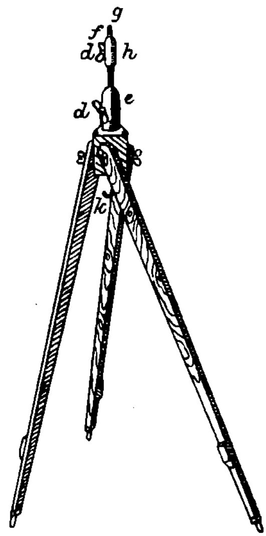

Fig. 1.

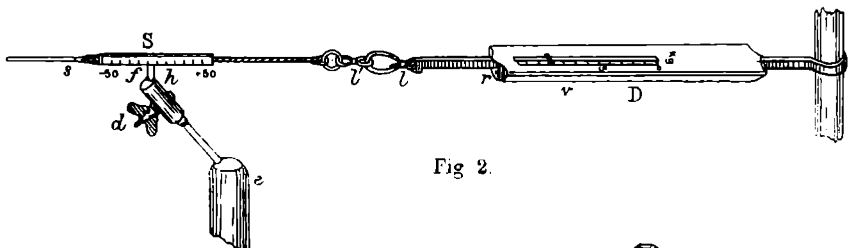

Fig 2.

Fig. 2a.

it may be, for example, hard drawn brass wire or phosphor bronze wire. The two wires should have the same diameter and should be silvered so as to present the same surfaces to the sun's rays.

Fine wires freely suspended in air take up almost exactly the temperature of the surrounding atmosphere: this is proved by the example of the diaphragm of a telescope directed to the sun: it is made of spider's web which is not burnt or even damaged although it is in the focus of the objective.

The slight alteration of temperature which the wires A and B undergo in the sun's rays has any how a very small effect especially if the measurement is made in the following way:—In the first section, that is from the first end of the base to the first support, measure first with A and then with B; in the second section from the first to the second support, first with B and then with A, and so on alternately: we can then assume the same temperature for the wires in the means of all the measurements. This temperature can be calculated from the following data:—The observation of the differences of length of the two wires during the operations on the ground, the coefficients of expansion and the difference of length of the two wires at the normal temperature determined by comparison with the standard.

It does not matter that the wires are stretched with a force of 10 kilogrammes instead of being at the normal tension, as long as they have been compared under this force of 10 kilogrammes. I have chosen a tension of 10 kilogrammes because it is sufficient to overcome the effect of wind and because, according to the way the spring-balance is used, it is approximately the muscular power of an ordinary man. [For the relation between the most convenient diameters of the wires and the tension of 10 kilogrammes *see* equation (27)].

Experiments made in 1879 show that it is better to use spring-balances than weights for the tension: the latter require more complicated arrangements on the ground and consequently take more time; and besides it is hard to get over the ill effects of the friction on the pullies over which the cords of the weights run.

The supports mentioned above and represented in fig. 1 ought to be placed 25 metres apart in the line of the base. After the support has been approximately fixed in the line of the base and its feet driven into the ground, the cylinder *f* which carries on its rounded top fine cross-marks at *g* can be adjusted more exactly both longitudinally and transversely by means of the ball and socket *e* which is clamped by the thumb-screw *d*. By means of a second ball and socket *h* the cylinder *f* can at the same time be made nearly vertical and placed so that one of the marks of the cross of reference is in the line of the base and the other at right angles to it. The feet of the supports should be placed two on one side and one on the other of the line of the base but never on it, so as not to interfere with the men working the spring-balances. When the tripods have to be placed on stony or rocky ground or in places where it is impossible to drive the feet into the ground, it is necessary to tie a weight to the hook *k*, such as a stone suspended in a net, in order to render the tripod stable.

To traverse obstacles of a certain size, such as a river or a deep depression of the land, wires of 50 metres are employed, but these should not entirely replace the 25 metre ones but should only be used in exceptional cases. Steel tapes are only employed for distances less than 25 metres, and besides the examination they should undergo in other respects, their division errors should naturally be taken into account.

The laying out of the supports in the line of the base can be done with the naked eye in ordinary cases, especially if little sights with cylindrical sockets to fit on the top of the cylinder *f* are used. If the error of aligument is *h*, that is, if a line drawn parallel to the line of the base and cutting the cross-marks of one of the supports passes at a distance h on one side of the other support, this will produce an error in the measurement which, according to equation (31), will be equal to

$$\frac{h^2}{2S}$$

where S is the distance between the supports. If S is 25 metres this error in the measurement of the length will be for

$$h = 1^{cm} \quad \ldots \quad \ldots \quad 0^{mm}.002$$
$$= 2 \quad \ldots \quad \ldots \quad 0\ .004$$
$$= 3 \quad \ldots \quad \ldots \quad 0\ .009$$

In favourable ground the alignment by eye does not introduce appreciable errors into the measurement of the length, provided always each support is placed in a straight line with the previous one and with a mark or signal fixed at a convenient distance on ahead or beyond the nearest terminal, so that the little error of alignment mentioned above is corrected *gradually* and not all at once.

In uneven ground the cylinder f (fig. 1) is placed carefully in a vertical position by means of a spherical level (fig. 2*a*) furnished underneath with a hollow cylindrical socket d' turned up square with the tangent plane of the level and which exactly fits over the cylinder f.

On the latter, the level is then replaced by a little telescope moveable on an axis which will be horizontal when its socket is fitted to the vertical cylinder f. This telescope is pointed forward on the alignment signal, and as it can be moved in a vertical plane it will serve to align the neighbouring support. The telescope is then transferred to the support next to that which has been adjusted, to determine the position of the one which comes after, and so on.

The progress of the work and the establishment necessarily depend on the following data:—First there are the two persons who place the supports. For that purpose they have a wire rope 25 metres long and provided at its two ends with simple balances fixed once for all to the rope, for the purpose of giving the tension of 10 kilogrammes. Near each extremity of the rope a short brass cylinder one centimetre long is soldered and provided with an index to mark the required length. This simple instrument like the tape and the wire is meant to hang freely suspended in air: it gives a provisional measure by means of which the cylinders f of the supports with the cross-marks are placed 25 metres apart. The first of these two persons has a workman to assist him in fixing the supports. Then come the leveller and his assistant in charge of the staff. This staff (called the "mire parlante" or staff for reading at sight) is 3 metres or more in height without any joint and having a level to place it vertical. It has two graduated scales one on each side, by means of which the levelling is controlled, *see* further on after equation (31). Lastly come the observers themselves. At the forward end of the wires there is a spring-balance, *see* D, (fig. 2), constructed with great care and submitted beforehand to a minute examination. Its construction is the same as the spring-balances usually found in the market, but it is more carefully examined and its details better finished. [For the study of the spring-balances see equation (34) and following]. It is at this extremity of the wires that the officer in charge reads the distances obtained at the index of the supports by means of the scales attached to the wires, *see* S, (fig. 2). He should watch carefully that the man in charge of the spring-balance, who should be an intelligent person, carefully avoids friction of the balance at r (fig. 2); he should watch that

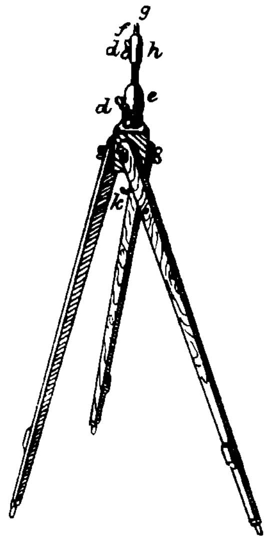

Fig. 1.

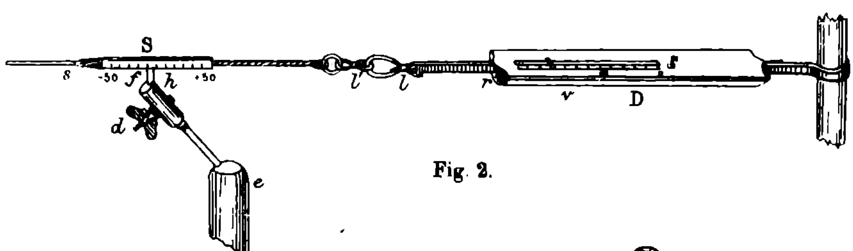

Fig. 2.

Fig. 2a.

the scale of the wire rests as lightly as possible on the cylinder of support *f*, that his assistant does not put his foot in the immediate neighbourhood of the feet of the support and lastly take care to place his eye so as not to produce any parallax in the observation of the index *v* on the scale of the spring-balance. The scale of the spring-balance can be adjusted and read to about 0·01 kilogrammes.

At the other end of the wire a more simple spring-balance is used, principally to counterbalance the force at the forward extremity. At the back end, the second observer, whose business it is to place the principal mark of the scale of the wire, that is the zero or middle mark, on the cross of reference of the support, can do so easily without leaning on the tripod when the tension is equal at the two extremities. If he does lean on the support he ought to do it as lightly as possible. To displace the scale of the wire a little in the direction of its length, an almost imperceptible force is sufficient just as a minimum of force is required to raise or lower one of the scales of a balance when they are equally loaded. The back observer who places the principal graduation mark of the wire on the index of the support calls out when the coincidence is exact. The man in charge of the front spring-balance should be able to say at the same moment that his is all right, that is to say at 10 kilogrammes. At the moment when both readings are correct, the officer in charge notes the reading of the forward scale of the wire on the index.

Reading glasses are fixed to the two cylinders *f* of the supports by means of the sockets on them so that the scales of the wires can be read exactly.

While the measurement with the wire A is going on two assistants hold the wire B ready, taking care that it does not touch the ground, *see* equation (30). The exchange of A and B is made by hooking on B to the spring-balance, after which the distance on the scale of this wire is noted as before.

It is well not to touch the two tripods in the rear of those on which work is going on, as they can be used again if one of those in front is slightly displaced.

To carry the tripods and the rest of the materials forward on the line of the base one more workman is necessary: this brings the personnel up to 12 of which 7 are workmen. If the work is portioned out in this way the men will be sufficiently occupied and will not get in one another's way.

Rapidity of observation without undue haste is recommended as the best way of avoiding everything which might lessen the accuracy of the work.

In this as in everything else it is not alone the instruments which give to the measurements the character of an operation of high accuracy but chiefly the attention given to everything which can influence them: this attention should be unceasing and based on a knowledge of scientific matters as well as on habits of observation. So that it is after all on the officer in charge that everything depends and it is he who can make the measurement of the base an operation of precision and get the greatest possible accuracy out of this method.

It is convenient to have 10 tripods which ought to be numbered so that the leveller and the person who notes the distances on the scales of the wires can easily verify the number of tripods employed. The numbers 5 and 10 should be painted a dark colour and the others light so that the numbers can be distinguished even at a distance from the levelling instrument.

The wires are rolled in bundles whose diameter is about 50 centimetres which corresponds to that which the wire takes up on coming out of the drawing-frame. The rolling and unrolling of the wire should be done with great care: the officer in charge should if possible do it himself or if not, a person accustomed to it.

If at any place on the wire a small bend is found, it may be presumed that its effect on the length, if sensible, will always be the same for the same tension and consequently it is unimportant. These bends should not therefore be eliminated without consideration for they ought to be left if they existed before the wire was measured against the standard.

The wire should be handled delicately. In particular, the assistants who carry the wire forward on the line should be cautioned to avoid every rough or jerky motion which might cause an accident: they ought indeed to try and diminish the shocks produced by walking by allowing the arms and the hand which holds the wire to give a little. It is important to remark that the point *s* (fig. 2) is a weak spot. It is there that the wire comes out of the socket where the scale is fixed. Unless care is taken the wire may easily be broken at this point. It ought never to be taken hold of by the scale but by the ring or by the chain at *l'*. It should always be held in a state of tension so that it may not touch the ground or curl up or twist, as kinks might subsequently be formed when it was restretched.

As a matter of fact experience proves, as will be shown later on, that in spite of their fragile appearance the wires do not undergo any alteration in their length in consequence of rolling and unrolling provided they are always handled carefully.

The scale S (fig. 2) should of course be placed vertically with the graduated edge below, flush with the cross of the cylinder *f*. For this purpose it must be possible to twist the wire through at least half a turn and to allow of this two swivel hooks *l* and *l'* are introduced. Experiments have been made to see how much the wires are shortened by this operation but no difference was found for 1, 2 or 3 turns. It would be easy to study theoretically this matter.

The graduated edge of the scale of the wire ought of course to be placed lightly at the exact point of the cross wires and along the mark of the line of the base.

Some of the recommendations and results of experience given above may seem useless but I think very few of them are really so, persuaded as I am that the accuracy of a measurement depends frequently and in a great measure on the combination of apparently small details.

Theory of Steel Tapes and Wires.

When a steel tape or metallic wire is suspended freely from two points it forms, if supposed absolutely flexible, a catenary given by the following equations :—

$$\frac{y}{k} = \frac{e^{\frac{x}{k}} + e^{-\frac{x}{k}}}{2} = 1 + \frac{1}{2}\left(\frac{x}{k}\right)^2 + \frac{1}{2.3.4}\left(\frac{x}{k}\right)^4 + \frac{1}{2.3.4.5.6}\left(\frac{x}{k}\right)^6 + \ldots \quad (1)$$

$$\frac{l}{k} = \frac{e^{\frac{x}{k}} - e^{-\frac{x}{k}}}{2} = \frac{x}{k} + \frac{1}{2.3}\left(\frac{x}{k}\right)^3 + \frac{1}{2.3.4.5}\left(\frac{x}{k}\right)^5 + \ldots \quad (2)$$

$$\frac{\rho}{k} = \frac{e^{\frac{2x}{k}} + e^{-\frac{2x}{k}} + 2}{4} = \left(\frac{y}{k}\right)^2 \quad \ldots \quad (3)$$

$$y^2 = k^2 + l^2 \quad \ldots \quad (4)$$

$$\mathrm{T} = \omega y \quad \ldots \quad (5)$$

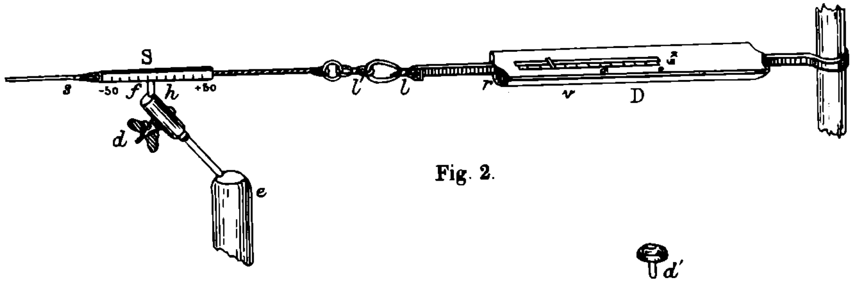

Fig. 2.

Fig. 2a.

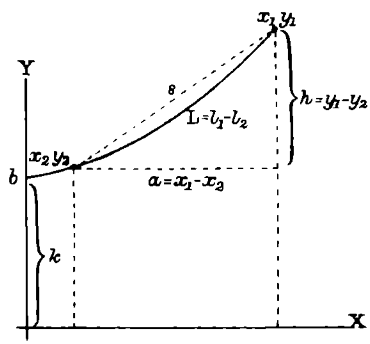

Fig. 3.

where the axis of y is vertical and the axis of x horizontal, the first passing through the lowest point of the curve b (fig. 3) and the second at a distance k (equal to the radius of curvature at b) below b: where also l is the length of the curve measured from the lowest point, ρ the radius of curvature, T the tension, ω the weight of unit length of tape and e the base of the Napierian System of logarithms.

Suppose now that the tape which has a length $L = l_1 - l_2$ is suspended between two points $(x_1\ y_1)$, $(x_2\ y_2)$, that the tension is T_1 at the first and T_2 at the second point and that the rectilinear distance between them is s: the horizontal projection of this line is $a = x_1 - x_2$ and the vertical projection is $h = y_1 - y_2$, that is, the difference of level determined by levelling to the two ends of the tape.

The constant quantities are L the length of the tape, h the difference of level, T_1 the tension at the point $(x_1\ y_1)$ and ω the weight of a unit length of tape.

The quantities required are $L - s$ or $L - a$.

Equation (2) gives

$$L = \frac{k}{2}\left(e^{\frac{x_1}{k}} - e^{-\frac{x_1}{k}} - e^{\frac{x_2}{k}} + e^{-\frac{x_2}{k}}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and equation (1) gives

$$h = \frac{k}{2}\left(e^{\frac{x_1}{k}} + e^{-\frac{x_1}{k}} - e^{\frac{x_2}{k}} - e^{-\frac{x_2}{k}}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

whence

$$L + h = k\left(e^{\frac{x_1}{k}} - e^{\frac{x_2}{k}}\right)$$

$$L - h = k\left(-e^{-\frac{x_1}{k}} + e^{-\frac{x_2}{k}}\right)$$

$$L^2 - h^2 = k^2\left(e^{\frac{a}{k}} + e^{-\frac{a}{k}} - 2\right)$$

therefore

$$\sqrt{L^2 - h^2} = k\left(e^{\frac{a}{2k}} - e^{-\frac{a}{2k}}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

Developing both sides we get

$$L - \frac{h^2}{2\,L} - \frac{h^4}{8\,L^3} - \frac{h^6}{16\,L^5} - \&c. = 2k\left[\frac{a}{2k} + \frac{1}{2.3}\left(\frac{a}{2k}\right)^3 + \frac{1}{2.3.4.5}\left(\frac{a}{2k}\right)^5 + \&c.\right]$$

so that

$$L - a = \frac{a^3}{24k^2} + \frac{a^5}{1920k^4} + \&c. + \frac{h^2}{2L} + \frac{h^4}{8L^3} + \&c. \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

We have also

$$s - a = s - \sqrt{s^2 - h^2} = \frac{h^2}{2s} + \frac{h^4}{8s^3} + \&c.$$

so that

$$L - s = \frac{a^3}{24k^2} + \frac{a^5}{1920k^4} + \&c. - \frac{h^2\,(L - s)}{2Ls} - \&c. \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

We can get from equation (1) the following expression for h instead of equation (7).

$$h = \frac{x_1^2 - x_2^2}{2k} + \frac{x_1^4 - x_2^4}{2.3.4.\ k^3} + \&c.$$

or

$$h = \frac{a\,(2x_1 - a)}{2k} + \&c.$$

and

$$h = \frac{a\,(2x_2 + a)}{2k} + \&c.$$

whence we get the following approximate expressions for x_1 and x_2

$$\left.\begin{aligned} x_1 &= \frac{k}{a}\cdot h + \frac{a}{2} \\ x_2 &= \frac{k}{a}\cdot h - \frac{a}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (11)$$

We must now get the value of k.

Equation (5) is equivalent to

$$\begin{aligned} k &= \frac{T}{\omega}\cdot\frac{k}{y} \\ &= \frac{T}{\omega}\cdot\frac{1}{1 + \frac{1}{2}\left(\frac{x}{k}\right)^2 + \&c.} \\ &= \frac{T}{\omega} - \frac{T}{\omega}\cdot\frac{x^2}{2k^2} \quad \ldots\ldots\ldots\ldots \quad (12) \end{aligned}$$

In ordinary cases $T = 10$ kilogrammes and $\omega = 0^{kg}.02$; so that for $x = 0$, $k = 500$ metres. The difference of level h is never greater than 3 metres while $a = 25$ metres; so that in the extreme case we have $x_1 = 72^m.5$ and $x_2 = 47^m.5$; so that the second term of the expression for k in (12) is only about the hundredth part of the first. In the other case the second term is still less. Thus k is very nearly constant and equal to the first term of (12) and therefore we may put $\frac{T}{\omega}$ for k in the denominator of the second term.

Thus

$$k = \frac{T}{\omega} - \frac{\omega}{T}\cdot\frac{x^2}{2} \quad \ldots\ldots\ldots\ldots \quad (13)$$

or approximately by means of (11)

and

$$\left.\begin{aligned} k &= \frac{T_1}{\omega} - \frac{h}{2} - \frac{T_1}{\omega}\cdot\frac{h^2}{2a^2} \\ k &= \frac{T_2}{\omega} + \frac{h}{2} - \frac{T_2}{\omega}\cdot\frac{h^2}{2a^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (14)$$

From (11) combined with (14) we have with sufficient approximation

$$\left.\begin{aligned} x_1 &= \frac{T}{\omega} \cdot \frac{h}{a} + \frac{a}{2} \\ x_2 &= \frac{T}{\omega} \cdot \frac{h}{a} - \frac{a}{2} \end{aligned}\right\} \quad \ldots \quad (15)$$

where T represents without sensible error either T_1 or T_2.

From this it follows that:—

$$\left.\begin{aligned} y_1 &= \frac{T_1}{\omega} + \frac{\omega}{T_1} \cdot \frac{a^2}{8} && = \frac{T_2}{\omega} + \frac{\omega}{T_2} \cdot \frac{a^2}{8} + h \\ y_2 &= \frac{T_1}{\omega} + \frac{\omega}{T_1} \cdot \frac{a^2}{8} - h && = \frac{T_2}{\omega} + \frac{\omega}{T_2} \cdot \frac{a^2}{8} \end{aligned}\right\} \quad \ldots \quad (16)$$

If $y_1 = y_2$ and consequently $h = 0$, that is to say, if the tape takes a horizontal position, the lowest point of the curve is at a depth p below the line s joining the terminal points determined by the equation

$$p = y_1 - k = y_2 - k$$

or from equations (14) and (16) by

$$p = \frac{\omega a^2}{8T_1} = \frac{\omega a^2}{8T_2}. \quad \ldots \quad (16')$$

If in (10) we put the value of k from (14) we get

$$\left.\begin{aligned} L - s = {} & \frac{a^3\omega^2}{24\,T_1^2} + \frac{a^5\omega^4}{1920\,T_1^4} + \&c. \\ & + \frac{a^3\omega^3}{24\,T_1^3} \cdot h + \frac{a\omega^2}{24\,T_1^2} \cdot h^2 + \&c. \\ & - \frac{L - s}{2Ls} \cdot h^2 - \&c. \end{aligned}\right\} \quad \ldots \quad (17)$$

where the terms neglected do not amount to $0^{mm}.0001$ in the ordinary cases.

In this expression for $L - s$, the unknown a ought to be replaced by its value in terms of the known L. This may be done by putting in the first term, which will almost always be the only important term, *see* equation (21) and following, its value from equation (9), *viz.*,

$$a = L - \frac{a^3}{24k^2} - \frac{h^2}{2L}$$

or

$$a = L - \frac{a^3\omega^2}{24T_1^2} - \frac{h^2}{2L} = L - \frac{L^3\omega^2}{24T_1^2} - \frac{h^2}{2L} \quad \ldots \quad (17')$$

and in the other terms $a = L$.

This gives

$$L - s = \frac{L^3\,\omega^2}{24\,T_1^{\,2}} - \frac{3L^5\,\omega^4}{640\,T_1^{\,4}}$$
$$+ \frac{L^3\,\omega^3}{24\,T_1^{\,3}} \,.\, h - \frac{L\omega^2}{48\,T_1^{\,2}} \,.\, h^2$$
$$- \frac{L - s}{2L\,s} \,.\, h^2.$$

The numerator L − *s* of the second term can be replaced with sufficient exactitude by the first term of the right hand side and the *s* of the denominator by L. We thus obtain

$$L - s = \frac{L^3\,\omega^2}{24\,T_1^{\,2}} - \frac{3L^5\,\omega^4}{640\,T_1^{\,4}} + \frac{L^3\,\omega^3}{24\,T_1^{\,3}} \,.\, h - \frac{L\omega^2}{24\,T_1^{\,2}} \,.\, h^2 \quad . \; . \; . \; . \; . \; . \; . \quad (18)$$

This difference between the tape hanging in a curve and the rectilinear distance between its extremities is called the *correction for curvature* and is denoted by $-c$; (c is always negative as a correction to the measured length). Its value is

$$c = -\frac{L^3\,\omega^2}{24\,T_1^{\,2}} + \frac{3L^5\,\omega^4}{640\,T_1^{\,4}} - \frac{L^3\,\omega^3}{24\,T_1^{\,3}} \,.\, h + \frac{L\omega^2}{24\,T_1^{\,2}} \,.\, h^2 \,. \quad . \; . \; . \; . \; . \; . \; . \quad (18')$$

We must now seek the *correction* c_1 *for increase of length due to tension.*

For this purpose we must introduce into the calculations the quantity σ which represents the amount that one metre of tape is stretched by a tension of weight ω. We find that σ is a constant since it is independent of the thickness of the tape (its sectional area) and only depends on the material employed, for ω is proportional to the area of the section. For steel σ is about 0·00000041 and for hard drawn brass 0·00000094.

For an element of length dl of the tape when the tension is T, the stretching is equivalent to

$$dc_1 = \frac{\sigma T}{\omega} \,.\, dl$$

or by means of (5)

$$dc_1 = \sigma y \,.\, dl$$

whence from (2)

$$dc_1 = \sigma y \,.\, \frac{e^{\frac{x}{k}} + e^{-\frac{x}{k}}}{2} \,.\, dx$$

or according to (1) and (3)

$$dc_1 = \sigma .\, \frac{y^2}{k}\, dx = \sigma\rho dx$$

$$= \frac{\sigma k}{4}\left(e^{\frac{2x}{k}} + e^{-\frac{2x}{k}} + 2\right) dx$$

whence

$$c_1 = \frac{\sigma k}{4} \int_{x_2}^{x_1} \left(e^{\frac{2x}{k}} + e^{-\frac{2x}{k}} + 2\right) dx$$

$$= \frac{\sigma k}{4}\left[4(x_1 - x_2) + \frac{4(x_1^{\,3} - x_2^{\,3})}{3k^2} + \quad . \; . \; . \;\right]$$

that is
$$c_1 = \sigma ka + \frac{\sigma(x_1^3 - x_2^3)}{3k} + \&c. \qquad (19)$$

Introducing the values of x_1 and x_2 from (15) and of k from (14) we get

$$c_1 = \frac{\sigma a T_1}{\omega} + \frac{\sigma\omega a^3}{12\,T_1} - \frac{\sigma a h}{2} + \frac{\sigma T_1}{2\omega}\cdot\frac{h^2}{a};$$

then putting in the value of a from (17′) we finally get

$$c_1 = \frac{\sigma L T_1}{\omega} + \frac{\sigma L^3\omega}{24\,T_1} - \frac{\sigma L h}{2} \qquad (20)$$

Thus then if we consider the corrections for curvature and tension, we find from equations (18′) and (20) that the rectilinear distance S between the extreme points of the tape is given by the following equation in which L_0 is the normal length, that is to say, the length we get on a plane surface without pulling or stretching:—

$$\left.\begin{aligned} S &= L_0 + c + c_1 \\ &= L_0 - \frac{L_0^3\omega^2}{24\,T_1^2} + \frac{\sigma L T_1}{\omega} + \frac{3L_0^5\omega^4}{640\,T_1^4} + \frac{\sigma L_0^3\omega}{24\,T_1} \\ &\quad - \left(\frac{L_0^3\omega^3}{24\,T_1^3} + \frac{\sigma L}{2}\right)h \\ &\quad + \frac{L_0\omega^2}{24\,T_1^2}\cdot h^2 \end{aligned}\right\} \qquad (21)$$

This is the *fundamental equation of our method of measurement.* In its application it must be observed that h is positive if the end of the tape at which the tension is T_1 is highest.

To get a better idea of the ordinary numerical values of the above terms we will make the following suppositions:—

$$L_0 = 25^{m} \qquad T_1 = 10^{k} \qquad \omega = 0^{k}.02 \qquad \sigma = 0{\cdot}0000004.$$

we then get

$$\begin{aligned} S = L_0 &- 2^{mm}.604 + 5^{mm}.000 + 0^{mm}.0007 + 0^{mm}.0005 \\ &- (0^{mm}.0052 + 0^{mm}.0050)\,h \\ &+ 0^{mm}.0042\,h^2 \end{aligned}$$

$$\text{or}\quad S = L_0 + 2^{mm}.397 - 0^{mm}.010\,h + 0^{mm}.0042\,h^2$$

if h is measured in metres.

We thus see that all the terms of the correction are negligible except the first two, that is the first term of each of the expressions for c and c_1 in equations (18′) and (20).

The sum of the two terms multiplied respectively by h and h^2 is for

$h = -3^{m}$	. .	$+0^{mm}.07$
$= -2$	. .	$+0$ .04
$= -1$	. .	$+0$ .01
$= 0$	. .	0
$= +1$	. .	-0 .01
$= +2$	. .	0
$= +3$	. .	$+0$ .01

These quantities can be neglected, for a difference of level of 2 or 3 metres will rarely occur.

Consequently we can put in almost every case

$$S = L_0 - \frac{L_0^3 \omega^2}{24T^2} + \frac{\sigma LT}{\omega} \quad \ldots \ldots \quad (22)$$

in which T is put instead of T_1 as there is no longer any necessity to distinguish the tensions at the ends of the tape.

Also in the expressions for c and c_1 in equations (18′) and (20) we can cut out all the terms except the first in each.

By the *normal tension* on page 3 we mean the tension which makes the rectilinear distance S between the two ends equal to the length L_0, which is the length of the tape on a plane surface without stretching. This tension called T_0 can be easily determined from equation (22) to be

$$T_0 = \omega \sqrt[3]{\frac{L^2}{24\sigma}} \quad \ldots \ldots \quad (23)$$

We can determine ω by weighing and σ by experiments in stretching.

When the whole length of the tape is not employed, as for example, on arrival at one of the ends of the base, and when we only use the fraction n so that the length is nL, the normal tension T_0^n is naturally different from that of the whole length.

Its value is from equation (23)

$$T_0^n = \omega \sqrt[3]{\frac{n^2 L^2}{24\sigma}} = T_0 \sqrt[3]{n^2}. \quad \ldots \ldots \quad (24)$$

Unlike steel tapes, metallic wires are always employed in their full length. An exact determination of T_0 is not therefore necessary since both in comparing against the standard and in field work the same T is used, and consequently S or the rectilinear distance between the principal marks on the scales will always be the same. Nevertheless it is necessary to know approximately the values of ω, σ, c, c_1, T_0, &c., to be able to apply to the resulting measure some small corrections which will be discussed further on. As stated above σ is almost constant for the same metal and its mean values are given approximately on page 12. It is necessary to add that ω is proportional to the area of the section or to d^2, where d is the diameter of the wire which can be measured by means of a micrometer screw, and to the specific gravity of the metal.

For approximate estimation we obtain without difficulty the following table in which, d being expressed in millimetres, ω is in kilogrammes per metre: the densities given below may be taken as correct.

	σ	Density.	ω	$\frac{\sigma}{\omega}$
Steel ...	0·00000041	7·8	$0{\cdot}00613\, d^2$	$0{\cdot}000067\, \frac{1}{d^2}$
Brass ...	0·00000094	8·5	$0{\cdot}00668\, d^2$	$0{\cdot}000141\, \frac{1}{d^2}$

	$L = 25^m$			$L = 50^m$
	c	c_1	T_0	T_0
Steel ...	$-\,24^{mm}.4\,\frac{d^4}{T^2}$	$+\,1^{mm}.67\,\frac{T}{d^2}$	$2^k.44\, d^2$	$3^k.88\, d^2$
Brass ...	$-\,29^{mm}.0\,\frac{d^4}{T^2}$	$+\,3^{mm}.52\,\frac{T}{d^2}$	$2^k.02\, d^2$	$3^k.21\, d^2$

. . . . . . (25)

It is not indispensable to keep to the force T_0 in measuring with wires, for the principle to follow should be to choose the force which makes the variations of S as small as possible with reference to small variations of T caused by a slight want of steadiness in the manipulation of the spring-balance. Differentiating equation (22) we get

$$\frac{dS}{dT} = \frac{L_0^3\, \omega^2}{12\, T^3} + \frac{\sigma}{\omega} L_0 \qquad \text{. . . . . (26)}$$

The relation between ω and T should be chosen so that this is a minimum. For reasons given on page 5, T has been fixed at 10 kilogrammes so we must seek the corresponding value of ω. The right hand side of (26) is a minimum if

$$\frac{L_0^3\, \omega}{6T^3} - \frac{\sigma L_0}{\omega^2} = 0$$

that is if

$$\omega = T \sqrt[3]{\frac{6\sigma}{L_0^2}}$$

or

$$T = \omega \sqrt[3]{\frac{L_0^2}{6\sigma}} = T_0 \sqrt[3]{4} = 1{\cdot}59\, T_0 \qquad \text{. . . . . (27)}$$

Comparing this result with equation (24) we see that T ought to be the normal tension for *double the length*, or if T is 10 kilogrammes, that ω should be chosen so that the normal tension for double the length of wire should be 10 kilogrammes, that is to say

$$\omega = 10 \sqrt[3]{\frac{6\sigma}{L_0^2}}.$$

According to the data of (25), ω should be chosen for wires of 25 metres so that

for steel ... $3^k.88 d^2 = 10^k$ or $d = 1^{mm}.61$

for brass ... $3^k.21 d^2 = 10^k$ or $d = 1^{mm}.77$

For the most of the wires employed up till now the diameter is about $1^{mm}.65$, *see* equation (40*a*) and following.

By equation (26) we have,

$$\left.\begin{array}{lll} \text{for} & T = T_0 & \dfrac{dS}{dT} = \dfrac{3\sigma}{\omega} L_0 \\ \text{,,} & T = T_0 \sqrt[3]{4} & \dfrac{dS}{dT} = \dfrac{3\sigma}{2\omega} L_0 \end{array}\right\} \quad \ldots \ldots \quad (27')$$

A little variation occuring in S caused by a little variation in T will therefore in the second case only be half what it would be in the first case.

For $T = T_0$ we have

$$S = L_0$$

but, for $T = T_0 \sqrt[3]{4}$ we have

$$S = L_0 + \frac{3}{4} \sqrt[3]{\frac{L_0^5 \sigma^2}{6}} \quad \ldots \ldots \quad (27'')$$

a value which is *independent of* ω.

A little error in the force given by the spring-balance will naturally entail a correction in S which can be deduced immediately from equation (26). In fact taking the data of equation (25) the corrections to the wires, according to equation (26), on account of faults of the spring-balance are the following, when T = 10 kilogrammes :—

$$\left.\begin{array}{lcc} & L = 25^{m} & L = 50^{m} \\ \text{Steel} & +\,0^{mm}.049\, d^4 + 1^{mm}.67 \dfrac{1}{d^2} & +\,0^{mm}.39\, d^4 + 3^{mm}.35 \dfrac{1}{d^2} \\ \text{Brass} & +\,0^{mm}.058\, d^4 + 3^{mm}.52 \dfrac{1}{d^2} & +\,0^{mm}.46\, d^4 + 7^{mm}.04 \dfrac{1}{d^2} \end{array}\right\} \quad \ldots \quad (27''')$$

[Appendix to equation (26)]

These corrections must be multiplied by those of the spring-balance expressed in kilogrammes.

Further, a correction dependent on the variation of gravity at different points of the earth's surface may sometimes be sensible: I mean the case in which the comparison with the standard and the measurement of the base are made in different places. It would then be necessary to distinguish tension by spring-balance and tension by weights.

1st. *Tension by spring-balances.* In what precedes, ω denotes the *weight* of wire per metre of length. If instead we denote *the mass* by m, the acceleration of gravity at the place of comparison by g and at the place of the operations by g', the weights will be respectively mg and mg': if besides the rectilinear lengths of the wire between the terminal points are denoted by S and S′ respectively, we have

$$S' = S + \frac{L_0^3 m^2 g^2}{24 T^2} \cdot \left(1 - \frac{g'^2}{g^2}\right)$$

or

$$S' = S + \frac{L_0^3 \omega^3}{12 T^2} \cdot \frac{g - g'}{g} \quad \ldots \ldots \quad (28)$$

2nd. *Tension by Weight* if it should happen to be employed. While in the preceding case ω underwent modification and T remained constant now they are both variable. If the mass of the weight is M, its weight at the place of comparison will be Mg and on the field of operations Mg′, which gives

$$S' = S - \frac{\sigma L T}{\omega} \cdot \frac{g - g'}{g} \quad \text{. . . . . (29)}$$

and according to Broch

$$g = 9^{m}.78062 + 0^{m}.05084 \sin^2 \phi - 0^{m}.00000192\ H$$

where ϕ is the latitude and H the height above sea level in metres.

When the wear and tear of the silvering produces a modification in the weight of the wire between comparison and field work so that ω changes, the resulting difference of length will be determined by

$$\frac{dS}{d\omega} = -\frac{L_0{}^3\omega}{12T^2}$$

or if all the weight $L\omega$ is denoted by V

$$\frac{dS}{dV} = -\frac{L_0 V}{12T^2} \quad \text{. . . . . (30)}$$

This correction as well as those obtained by equations (28) and (29) will however hardly ever be perceptible. Nevertheless great care should be taken that the wire should not touch the ground during measurement as the dust which would adhere to it might make a modification in ω.

The distance S being known we must find its horizontal projection or the reduction K (always negative) to the horizontal distance. This is $K = S - a$ or

$$K = \frac{h^2}{2S} + \frac{h^4}{8S^3} + \frac{h^6}{16S^5} + \&c., \quad \text{. . . . . (31)}$$

where as before h is the difference of level between the top surfaces of the cylinders of the tripods where the cross-marks are drawn. This correction is got from a table prepared for the purpose.

A single error in reading the staff in the ordinary case affects two successive differences of level, and the corresponding values of K are wrong and also the final result of the whole measurement. An unintermitting control is therefore necessary and it is best obtained by providing the staff with two scales, one on each side, their divisions being painted in black on the one and in red on the other. The two scales are read by means of a level at each support and at each of the ends of the base. In order that the control may be fully effective, it is necessary that the two divisions should be so different from each other that one cannot calculate involuntarily in advance the reading which will be got on the opposite scale after having noted the first one. When, for example, the divisions are so disposed that the sum or the difference of the readings is constant, an experienced computer will have difficulty in keeping himself from deducing involuntarily and in advance the result that he will obtain on the second scale. To obviate this inconvenience the unit of the black scale is a decimetre and of the red $\frac{1}{7}$ of a metre. On the two scales the readings can be made to the $\frac{1}{100}$ part of the respective units and in both the readings increase from below to above, that is to say, the readings should be smaller when the point where the staff is placed is higher. The black scale commences below at 0 and the red at say 2·63 or any other figure.

The graduation of the scales to different units presents the following advantage which is not without importance. When one makes a mistake it is mostly of a whole unit (1 decimetre). If the scale is graduated to the same unit it is generally impossible to decide on which side the mistake was made. But on the other hand if the unit of one of the scales is a decimetre and of the other $\frac{1}{7}$ of a metre, a mistake of a unit in the first will correspond to a mistake of 0·7 on the second and a mistake of a unit in the second will correspond to a mistake of 1·43 in the first. With a single unit it is quite possible that one might be deceived, but a mistake of reading of 0·7 or 1·43 is hardly possible.

According to equation (31), we have, if h is the difference of level in decimetres and d the difference in units of $\frac{1}{7}$ of a metre, K being given in millimetres and S = 25 metres

$$K = 2\cdot 10^{-1} h^2 + 8\cdot 10^{-7} h^4 + 64\cdot 10^{-13} h^6 + \&c.,$$

and

$$K = \frac{d^2}{2\cdot 45} + \frac{d^4}{300125} + \frac{d^6}{18382656250} + \&c. = I + II + III, \&c.,$$

$$\text{where} \quad I = \frac{d^2}{2\cdot 45}; \qquad II = \frac{I^2}{50000}; \qquad III = \frac{I \times II}{25000}; \ \&c.$$

The two first terms only are sensible. The values of K are got from a table with d as the argument.

The control aimed at consists then in this, that the values of K obtained by one or other of the tables each with its own argument ought to agree with each other.

The terminal length on the arrival at the other end of the base, or more often the intermediate intervals when accidents of ground are met with, are generally less than 25 metres so that it follows that the tables prepared for these cases are useless. The calculation is most easily performed by means of the formula

$$\log a = \frac{\log (S + h) + \log (S - h)}{2} \quad \dots\dots\dots \quad (31')$$

If the division errors of the staff are sensible, a correction to K will result which will be very difficult and laborious to calculate if the division errors are irregular. But if the divisions are regular, we can express by $h(1 + f)$ a difference of level h determined by the scale of the staff and the corrected value K′ of the reduction to the horizontal distance will be

$$K' = \frac{h^2}{2S}(1 + f)^2 + \frac{h^4}{8S^3}(1 + f)^4 + \&c.$$

or with sufficient approximation

$$\left.\begin{aligned} K' &= (1 + 2f)\, K \\ \text{and } \Sigma K' &= (1 + 2f)\, \Sigma K \end{aligned}\right\} \quad \dots\dots\dots \quad (32)$$

By differentiating equation (31) we get

$$\frac{dK}{dS} = -\frac{h^2}{2S^2} = -\frac{K}{S} \quad \dots\dots\dots \quad (33)$$

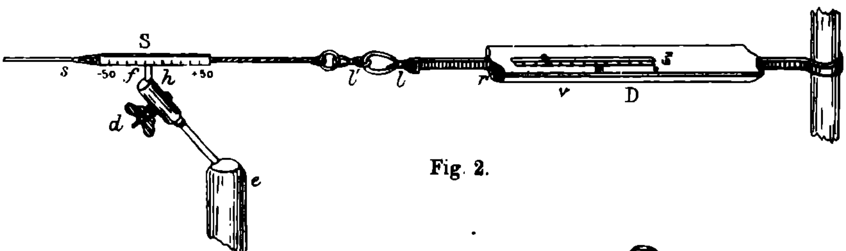

Fig. 2.

Fig. 2a.

It follows that when the rectilinear distance between the cross-marks of the tripods is not exactly 25 metres, it is necessary to apply a small correction which is negative as a correction to K but positive as a correction to the measured line, if the distance between the tripods is greater than 25 metres. If the correction for the length of the wire obtained by comparison with the standard is very small, the difference from 25 metres is immediately given by the reading e on the scale S of the wire (fig. 2), for the scales of the wires, each a decimetre long, are so graduated that the middle mark reads 0, the furthest from the other end of the wire $+ 50^{mm}$ and the nearest -50^{mm}.

The normal length of the wire (25 metres) is indicated by the rectilinear distance between the zeros of the scales at a tension of 10 kilogrammes and at a temperature of $+15^\circ$ C. It is thus the reading e which must be multiplied by the factor $-\frac{K}{S}$ to obtain the correction in question which is given in a table for $S = 25$ metres.

The correction is only perceptible in the case where the difference of level of the tripods is great and at the same time the tripods badly placed, that is to say, when they are placed at an interval which differs sensibly from 25 metres. Even in this case the correction never surpasses a small fraction of a millimetre.

The Spring-Balance.

The following formula corresponds to a spiral steel spring:—

$$f = k \frac{R^3 h}{d^4} \qquad \text{. . . . .} \qquad (34)$$

f being the lengthening of the spiral for a change of 1 kilogramme, R the mean of the exterior and interior diameters of the spiral, h the number of turns, d the diameter of the wire and k is a constant depending on the material employed. If R, d and f are given in millimetres we obtain as a mean for steel

$$k = 0{\cdot}0076.$$

This quantity is absolutely the same before and after tempering, an operation which therefore only changes the limits of elasticity and not the elasticity itself.

The choice of the kind of steel is of the greatest importance. The trials I have made on a great number of spirals of different steels, for the most part English, have proved that they have in general a fault which renders them quite unfit for the end in view. In fact, if we suspend the spring-balance vertically and attach a weight of 10 kilogrammes for example, we obtain immediately on the scale a reading which at the end of a minute will have considerably increased (by about $0^k.1$ to $0^k.3$) and will increase a little more in the following minutes. It is only after about 5 minutes that this reaction ceases and the reading remains constant.

A consequence of this fact is that the reading of the spring-balance is different for a certain pull, according as this pull has been increased or decreased immediately before the operation: it is less in the first case and greater in the second.

The same kinds of steel have often also the fault that the stretching f per kilogramme, *see* equation (34), increases with the pull, which gives a want of uniformity to the scale of the spring-balance. In spite of the numerous attempts to harden the steel these inconveniences have always remained in certain qualities of the metal. Up to the present I have not been able to find any kind really good in this respect except that from the manufactory of Gunnebo in Sweden.

Generally speaking this steel does not show in the least the faults mentioned above. It has also proved itself excellent in other points of view, since spring-balances of this metal made in 1878 have not up to date varied sensibly and continue to have the same corrections they had 17 years ago.

A spring-balance graduated to mark the force in a horizontal direction cannot naturally be immediately used for ordinary weighing in a vertical direction, for there exists a constant difference between the readings obtained in the two positions for the same tension. This difference consists of the weight of the bar which comes out of the hole at r (fig. 2) with the swivel l attached to it, as well as the pointer v augmented by half the weight of the spring. If therefore the total weight of these different parts is denoted by y and if the spring-balance with a pull of 0 gives a correction x in the horizontal position, it follows that the reading a in the vertical position without pull will be

$$a = -x + y$$

If we then suspend the spring-balance turning it downwards, the reading a' will evidently be the following, where D is the total weight of the spring-balance

$$a' = D - x - y,$$

which gives

$$\left.\begin{aligned} x &= \frac{D - a - a'}{2} \\ y &= \frac{D + a - a'}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (35)$$

We can immediately note a and a', then weigh the spring-balance and find x and y. The last correction y can also be obtained more exactly by placing on a balance the different parts of the instrument which we have indicated above.

After the difference of the readings in the vertical and horizontal positions has been found the spring-balance is proved, suspended vertically, by means of weights. This proof leads to a correction C, in the horizontal position, expressed by

$$C = x + m_1 T + m_2 T^2$$

where x is as before, m_1 and m_2 constants and T the pull. For a good spring-balance m_2 should be $= 0$ and x and m_1 very small. We must however recollect that these terms vary with the temperature and that this variation is always sensible in m_1.

As I have said above, on page 5, the distances are noted alternately with the wires A and B in such a way that at every second tripod A is used first and then B and at the intermediate measurements B first and then A. It is therefore necessary at each measurement to detach the spring-balance from one of the wires and attach it to the other. To save time two pairs of spring-balances can be used, one of which is of the superior kind and the other the medium kind: these will be fitted one of each kind to the extremity of each wire and left there.

At the extremity of the wires where the readings are made and where the good spring-balances are placed, it is moreover necessary to make systematic exchanges between them every five measures say, for if the wire A is always joined to the same spring-balance and likewise the wire B, we could not count on the advantage in knowing that the errors of the spring-balances are eliminated, *see* (40*a*) and following.

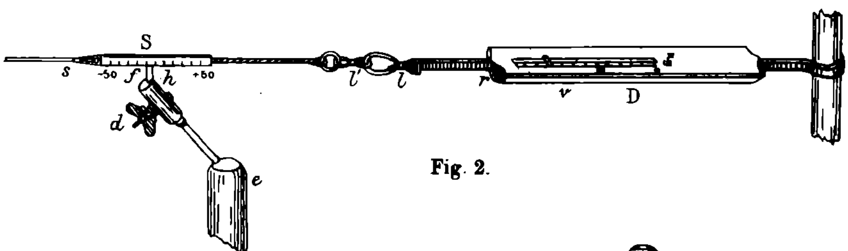

Fig. 2.

Fig. 2a.

Naturally it is necessary to provide oneself with more than one pair of wires 25 metres in length in case an accident happens to the pair A and B.

Calculation of the Base Line.

Suppose that the wire A freely suspended and stretched with a force of 10 kilogrammes has a length (the rectilinear distance between the zeros of the scales) which is S_a at the temperature of comparison, (+ 15° C.) and which is expressed at any other temperature by

$$S_a\,[1 + \alpha_1\,(t - 15^\circ) + \alpha_2\,(t - 15^\circ)^2]$$

and that similarly the length of the wire B is expressed by

$$S_b\,[1 + \beta_1\,(t - 15^\circ) + \beta_2\,(t - 15^\circ)^2]$$

t being the temperature.

Supposing then that e_a and e_b are the readings on the scales of A and B respectively and n the number of intervals between tripods, the length λ of the broken line which passes over the heads of the supports will be:—

$$\left.\begin{array}{l}\text{For wire A} \ldots \lambda = nS_a + \Sigma e_a + nD_a + \alpha_1 nS_a\,(t - 15^\circ) + \alpha_2 nS_a\,(t - 15^\circ)^2 \\ \quad ,, \quad ,, \quad \text{B} \ldots \lambda = nS_b + \Sigma e_b + nD_b + \beta_1 nS_b\,(t - 15^\circ) + \beta_2 nS_b\,(t - 15^\circ)^2\end{array}\right\} \quad . \quad . \quad . \quad (36)$$

nD_a and nD_b being the sums of the corrections arising from errors of the spring-balance [equation (26) and the data of (27''')] and, if they are sensible, of those depending on equations (28) and (30). These two expressions for the length of the broken line should be equal and we get, putting

$$\left.\begin{array}{l} nS_a + \Sigma e_a + nD_a = A \\ nS_b + \Sigma e_b + nD_b = B \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36')$$

the following expression by means of which the temperature should be determined:—

$$t - 15^\circ = \frac{A - B - A\,(\beta_2 - \alpha_2)\,(t - 15^\circ)^2}{A\,(\beta_1 - \alpha_1)}$$

A having been introduced with a sufficient degree of approximation in the little terms to represent either of the quantities nS_a or nS_b which are almost the same. Finally we obtain for the determination of the temperature

$$t - 15^\circ = \frac{1}{\beta_1 - \alpha_1} \cdot \frac{A - B}{A} - \frac{\beta_2 - \alpha_2}{(\beta_1 - \alpha_1)^3}\left(\frac{A - B}{A}\right)^2 + 2\,\frac{(\beta_2 - \alpha_2)^2}{(\beta_1 - \alpha_1)^5}\left(\frac{A - B}{A}\right)^3 - \&c. \quad (37)$$

of these terms the two first at the most are perceptible, for we have about:—

For Steel	$\alpha_1 = 0{\cdot}000010$		$\alpha_2 = 0{\cdot}000000005$ (?)
,,	Brass	$\beta_1 = 0{\cdot}000018$	$\beta_2 = 0{\cdot}000000010$

the temperature being reckoned in degrees centigrade. These co-efficients of expansion α_1 and β_1 are less than for ordinary steel and brass. In fact the expansion is smallest when the metal is rolled or hard drawn.

If we introduce the value of $t - 15°$ from equation (37) into equation (36) we get

$$\left.\begin{aligned}\lambda &= A + \frac{a_1}{\beta_1 - a_1}\cdot(A - B) + \frac{a_2\beta_1 - a_1\beta_2}{(\beta_1 - a_1)^3}\cdot\frac{(A - B)^2}{A}\\ \lambda &= B + \frac{\beta_1}{\beta_1 - a_1}\cdot(A - B) + \frac{a_2\beta_1 - a_1\beta_2}{(\beta_1 - a_1)^3}\cdot\frac{(A - B)^2}{A}\end{aligned}\right\} \quad \ldots\ldots \quad (38)$$

Thus we see that the term proportional to $(A - B)^2$ can become zero even if in the expansions the term proportional to the second power of the temperature is sensible, *viz.*, when $a_2\beta_1 = a_1\beta_2$.

If in the determination of the expansions, the search is limited to finding the terms proportional to the first power of the temperature, we should naturally put

$$\left.\begin{aligned}\lambda &= A + \frac{a}{\beta - a}\cdot(A - B)\\ &= B + \frac{\beta}{\beta - a}\cdot(A - B)\\ &= \frac{\beta}{\beta - a}\cdot A - \frac{a}{\beta - a}\cdot B\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (39)$$

with the mean approximate values given above for a_1 and β_1

the coefficient $\frac{a_1}{\beta_1 - a_1} = 1\cdot25$ and $\frac{\beta_1}{\beta_1 - a_1} = 2\cdot25$.

Another way of finding the temperature and determining the length of the base-line which lends itself better perhaps to numerical calculation is the following:—From the comparison with the standard we get the difference of length $S_a - S_b$ of the wires at the fundamental temperature + 15° C. In addition their expansions are known (either only the first term or both), so that their difference of length is known at any other temperature and a table can be made to give this within the limits of temperature that can arise. Now we know from the field-book what are the values of Σe_a and Σe_b, that is, the sums of the readings on the scales of the wires, and it is easy to find the temperature with the mean difference of the readings $\frac{\Sigma e_a - \Sigma e_b}{n}$ as argument by interpolation from this table. For each interval between two successive tripods the difference of reading of the two wires ought to be nearly constant since the temperature does not vary rapidly, and this circumstance in addition provides the necessary control for avoiding gross errors in the readings. These differences will therefore be noted each time in the field-book. Their mean is really that mentioned above although calculated in another way.

After the temperature is found, λ is calculated from the *two* expressions of (36). A sure and sufficient control of this calculation consists in the fact that the two values, leaving out errors of observation, ought to agree *exactly*.

The corrections nD_a and nD_b occurring in the expression for λ in equation (36) in general only take account of the corrections caused by faults in the spring-balances [see equations (26) and (27''')]. If A' and B' denote A and B *without* these corrections so that by (36')

$$A = A' + nD_a$$

$$B = B' + nD_b$$

the first of equations (39) becomes

$$\lambda = A' + nD_a + \frac{a}{\beta - a}(A' + nD_a - B' - nD_b)$$

or

$$\lambda = A' + \frac{a}{\beta - a}(A' - B') + \frac{n}{\beta - a}(\beta D_a - aD_b). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

The error committed in the measurement of the base-line, owing to the neglect of the corrections to the spring-balance is given therefore by the last term of equation (40). This error is zero if $\beta D_a = aD_b$, that is according to (26) if

$$a : \beta = \frac{dS_a}{dT} : \frac{dS_b}{dT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40_a)$$

Consequently we can neglect the correction for the spring-balance, or in other words the influence of the error of the spring-balance is compensated, if the coefficients of expansion are proportional to the corrections of the wire which depend on the errors of the spring-balances.

It is clear however that the correction for the spring-balance should be taken into account *in the comparison with the standard.* The last proposition is therefore only valid if the temperature is determined by the difference of the lengths of the wires, and this is not usually done in the comparisons where thermometers are used.

As the diameters d are equal for the two wires and as the co-efficient of expansion is for steel 0·000010 and for brass 0·000018 this diameter according to (27''') should be chosen so that

$$0{\cdot}000018\left(0{\cdot}049\,d^4 + \frac{1{\cdot}67}{d^2}\right) = 0{\cdot}000010\left(0{\cdot}058\,d^4 + \frac{3{\cdot}52}{d^2}\right)$$

or

$$d = 1^{mm}.60$$

in order that the condition (40_a) may be satisfied.

This is the same value of the diameter found on page 15 to be that which with a tension of 10 kilogrammes gives a minimum for the variations of the length of the wire corresponding to variations of the spring-balance. Since the wires generally in use up to the present have a diameter about $1^{mm}.65$ they very nearly fulfil the condition expressed in (40_a). Thus then the errors of the spring-balances influence only very slightly the result of the measurement.

After the broken line which passes over the heads of the supports has been calculated, it still remains to find the reduction to the horizontal distance [equations (31), (31') and (33)] and lastly, if they are sensible, the corrections from (32) for the two scales of the levelling staff.

We get the error in the measurement of the length arising from an error of levelling from the equation (31):—

$$-\frac{dK}{dh} = -\frac{h}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40')$$

The same error may be estimated otherwise by means of table I giving K. Up to $\frac{1}{4}$ metre of difference of level ($h = 2^{dm}.5$), an error of levelling of 1^{mm} introduces an error in K of $0^{mm}.01$ only: for $h = 25$ decimetres (the greatest difference of level which could occur) the same error of levelling produces an error in K of $0^{mm}.1$. Evidently we need not fear, in passable ground, the errors which may arise in careful levelling, if we take care before everything that the level of the telescope is verified immediately before each reading of the staff.

When the temperature is constant during the measurement, the differences of readings on the scales of the two wires ought also to be constant, and any variations which arise should be attributed to errors of observation. If in such a case it appears from the field-book that a difference of reading has a probable error $\pm r$, that evidently arises from a probable error $\pm \frac{r}{\sqrt{2}}$ made in a single reading.

According to the third of equations (39) we have now the probable error ϕ of all the broken line λ:—

$$\phi = \pm \frac{r}{1 - \frac{a}{\beta}} \sqrt{\frac{n}{2}\left(1 + \frac{a^2}{\beta^2}\right)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (41)$$

n being the number of intervals between the supports, each usually 25 metres, neglecting the probable error made in the last interval at the end of the line which is less than 25 metres and is measured with a steel-tape. If the temperature is very variable naturally r cannot be determined in this way, but if on the other hand the variations of temperature are insignificant as is generally the case, the variations of the differences depend *principally* on errors of observation, and the value r which is deduced from these observations will be a little too great, so that the probable error of the whole line λ will be smaller than that given by equation (41).

All this supposes that the comparison with the standard is free from error, that a and β are exactly known, that the expansion can be expressed by a single term proportional to the temperature and in short that for this reason formulæ (39) can be used instead of those of (38).

With the mean expansions for steel and brass given on page 21 ($a = 0{\cdot}000010$ and $\beta = 0{\cdot}000018$) we have according to (41)

$$\phi = \pm 1{\cdot}82 r \sqrt{n} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (41')$$

We see by (41) that ϕ will be the smaller the less $\frac{a}{\beta}$ is, that is the more difference there is between the expansions of the two metals.

At present it appears difficult to find two metals having more different expansions than those of steel and brass and at the same time possessing their good qualities. Phosphor bronze has several times replaced brass in the measurement of bases by means of wires. The expansion of this alloy is however no greater than that of brass. It is said that 'aluminium steel' possesses a considerable expansion and that it surpasses ordinary steel in tenacity but I am not sufficiently informed on the subject.

General Remarks on the Measurement of Bases by means of Metallic Wires.

The possibility of employing this method for measuring geodetic bases depends on the answers that we can give to the following two principal questions:—

1st. Is the temperature indicated with a sufficient degree of precision by the differences of lengths of the wires? In other words do the two wires possess under all circumstances, even in the direct rays of the sun, the *same* temperature, in the mean, for all the measurement?

2nd. Are the lengths of the wires themselves invariable? Or, if they are variable, does the modification of length act in a way favourable or unfavourable to this method of measurement, regularly or irregularly, quickly or slowly, independently or not of the rolling and unrolling of the wires and lastly is it influenced by the carriage and work in the field?

The best way to answer the first question is to measure the same line under different conditions at different times. This proceeding has afforded very satisfactory results and as an example I give the measures I have taken of a short base near Stockholm. They will be found in my memoir as an appendix to the *Revue de l' Académie royale des sciences de Suède* under the title *Geodätische Längenmessung mit Stahlbändern und Metalldrähten.* The two stations at the ends of the base are called **A′** and **B′** and the two intermediate stations **I** and **II**. The following were the results:—

Line A′ — I

1882.	8 May	Sun, scattered clouds	$739^{m}.7708$
	11 May	Sun, calm	·7723
	11 May	Wind	·7695
	12 May	Heavy rain and wind	·7725
		Mean =	$739·7713 \pm 0^{mm}.5$

Line I — II

1882.	9th May	Sun, strong wind	$367^{m}.0127$
	9th May	Calm	·0120
	10th May	Sun	·0079
	10th May	Sun	·0100
		Mean =	$367·0106 \pm 0^{mm}.7$

Line II — B′

1882.	9th May	Light clouds	$888^{m}.2372$
	10th May	Sun, light wind	·2371
	11th May	Sun, wind	·2287
	15th May	Sun, hurricane at right angles to line of base	·2380
		Mean =	$888·2352 \pm 1^{mm}.5$

I instance also in connection with this question the *Mesures des bases de Moloskovitzi et de Poulkovo exécutées en 1888 avec l'appareil de Jäderin* by the Russian General *A. Bonsdorff.* The base between Moskovitzi and Osertizi in the province of Ingermanland is 9822 metres long and is divided into six parts by five intermediate stations. The measures executed twice in the same divisions of this base agreed with each other although made at different times.

I have answered the second question in a little memoir also given in an appendix to the *Revue de l'Académie royale des sciences de Suède* entitled *Märklig längdförändring hos geodetiska basmätningssträngar* (Remarkable variation of the length of metallic wires for measuring geodetic bases). Although this memoir was only provisional; my intention being to undertake more complete researches on the subject, a short resumé of its contents may find a place here. In fact the question of the variation of wires is of the highest importance.

It happens that in the time which immediately follows their drawing, the wires have a tendency to shorten, due probably to a predisposition which the molecules have to take up the position which they occupied before the energetic treatment to which they were submitted. This shortening may be as much as 5 millimetres for a wire of 25 metres and takes place probably in the first months which follow drawing. At the end of six months the shortening seems to have ceased.

But the length of the wire is not constant for all that after this epoch, because new shortenings or lengthenings may take place alternately. What is necessary to remember here before all is that the wires do not undergo the slightest modification by reason of rolling up or transport (see for example the work of General Bonsdorff already quoted, page 25). A modification resulting from this cause would be of an extremely grave character.

The variations of the wire are of a very singular nature and their physical cause is unknown to me. In fact they all vary equally, lengthening or shortening themselves simultaneously and by approximately the same quantities. Those of brass and those of iron behave in the same way whether they are in use or not, whether kept stretched or rolled, carried about or left in one place, &c.

In 1879 I tried for the first time to measure bases with steel tapes and in 1882 with wires. They were compared with the standard by measuring a base of 100 metres marked in the garden of the Polytechnic School of Stockholm. The line was measured under different conditions by means of an apparatus belonging to the Academy of Sciences of Sweden, an apparatus which is a copy of that of Struve at Pulkowa. The apparatus consists of four measuring bars and two standards, all contact bars. The first are provided at one of their extremities with sensitive levers and the second have both ends fixed. One of the standards was compared in 1861 at Pulkowa by Lindhagen, Wagner and Döllen with the standard of the Russian apparatus (the double Toise of Struve). This Swedish standard has remained constantly at Stockholm since then, while the other has been carried about for comparing the four measuring bars. Nevertheless the difference between the two standards has remained constant during the thirty years which have elapsed, within the probable errors of observation. On the other hand the four measuring bars have altered, evidently on account of the sensitive levers.

From measures I took on the line of 100 metres with the wires from March 1882 to November 1883, I at first drew the conclusion that the base-line had altered during the time but that the wires remained in a constant state, for the measures made on the same day and under different conditions with from two to six wires, gave the same result. Otherwise I should have had to make "the very improbable supposition that all the wires altered in an absolutely identical way." It is however, as I have said before, this improbable circumstance which is the fact.

In support of my statement I give here first, from among the many standard comparisons made in the course of years on this base of 100 metres and by means of these six wires, only those which were made near the same time that it was measured with the apparatus of the Academy (the four measuring bars of this apparatus having been each time compared with the normal).

Wire	Metal	Epoch. 1883·88.	1884·74.	1886·45.	1893·45.	1895·45.
		m mm	mm	mm	mm	mm
A	Steel . . .	25 + 8·87	+ 9·10	+ 9·18	+ 8·30	+ 8·51
B	Brass . . .	+ 21·78	+ 22·60	+ 22·29	+ 21·16	(+ 20·45)
C	Brass . . .	− 1·60	− 1·69	− 0·89	− 1·77	− 1·62
D	Steel . . .	− 13·77	− 12·73	− 12·49	− 13·04	− 12·86
E	Steel . . .	− 3·75	− 3·63	− 3·01	− 3·58	− 3·31
F	Brass . . .	− 4·86	− 4·66	− 4·01	− 4·56	− 4·51

The intermediate times are here one or several years and yet there is among the wires a very manifest tendency to vary in company.

On the other hand in the following table the intermediate times are short. The wires A_{29} and C_{31} (steel) and B_{30} and D_{32} (brass) belong to the Russian Staff, while A to F are the same as above:

Wire	July 1888. 29	30	31	31	August 1888. 1	2	3	6
	m mm	mm	mm	mm	mm	mm	mm	mm
A_{29}	25 −0·39	−0·63	−0·69	−0·57	−0·55	−0·82	−0·85	−0·76
B_{30}	+0·58	+0·44	+0·50	+0·57	+0·61	+0·44	+0·39	+0·37
C_{31}	+0·20	−0·10	−0·15	−0·05	−0·03	−0·25	−0·36	−0·40
D_{32}	−0·67	−0·85	−0·80	−0·74	−0·62	−0·89	−0·93	−0·97
A	...	...	...	...	...	+10·04	+10·19	...
B	...	...	...	...	...	+22·88	+22·86	...
C	...	...	...	...	...	−0·37	−0·53	...
D	...	...	...	...	...	−11·88	−11·96	...
E	...	...	...	...	...	−2·30	−2·44	...
F	...	...	...	...	...	−3·48	−3·51	...
Steel Tape . .	...	...	...	...	+4·93	...	+4·64	...
Temperature.	+17°	17°	17°	18°	18°	17°	17°	21°

Wire	August 1888. 6	13	14	29	30	30	Sept. 1888. 1	4
	m mm	mm	mm	mm	mm	mm	mm	mm
A_{29}	25 −0·71	−0·90	−1·06	−1·00	−0·91	−1·09	−0·99	−0·98
B_{30}	+0·53	+0·35	+0·22	+0·13	+0·16	+0·10	+0·12	+0·10
C_{31}	−0·36	−0·67	−0·80	−1·29	−1·31	−1·39	−1·36	−1·37
D_{32}	−0·74	−0·87	−0·94	−1·01	−1·00	−0·98	−0·98	−0·97
A	...	...	...	...	...	...	+10·02	...
B	...	...	...	...	...	...	+22·57	...
C	...	...	...	...	...	...	−0·34	...
D	...	...	...	...	...	...	−12·09	...
E	...	...	...	...	...	...	−2·66	...
F	...	...	...	...	...	...	−3·81	...
Temperature.	+21°	16°	16°	16°	16°	17°	16°	16°

The comparison was made as I will describe further on, in the riding-school of the first Cadet corps of Vasili Ostrov at St. Petersburg, a place particularly adapted to such work. The nearness of the comparison is estimated at $\pm 0^{mm}.04$.

The variations of the wires are apparent in this case also, though they have been treated differently. From the 28th July to 10th August they remained stretched against one of the walls of the riding-school. On the 10th August the Russian wires were taken into the field and the pair B_{30} and C_{31} was employed to measure the great base of Pulkowa while the Swedish wires A to F remained suspended in the riding-school. On the 13th and 14th August the Russian wires were again suspended, after which both lots were taken to the field till the 29th August when they were brought back to the riding-school. In this interval B_{30} and C_{31} only were used daily on the ground and they were carried daily in company with A_{29} and D_{32} on a very jolting cart.

One would be very much inclined to suppose that the law of the variations we are now examining is of the same nature as that which directs the variations of zero of a thermometer. We wish to point out that, when the remarkable shortening which takes place after the manufacture has had its full effect, the variations should depend on the temperature in which the wires are kept immediately before comparison as compared with that which exists during the operation. I have not been able to make direct experiments to study this phenomenon, but the above table contains certain data which ought to throw light on it. In fact the temperature of the riding-school was almost constant from the 28th July to 3rd August 1888. The variations for this lapse of time did not exceed 1°. It appears then that the lengths themselves should have been constant during this time, for all the wires had been drawn several years before. From the 3rd to 6th August the riding-school was heated up in order to determine the expansions and then it was brought back to its original temperature. No change whatever could be found in consequence of this experiment.

Consequently for the present I regard the cause of the variations of the wires as absolutely unknown and I am content, for want of something better, to prove that the variations do not depend either on the employment of the wires in the field or on their transport: that the variations are not produced all at once but continuously, so that their lengths can be obtained by interpolations between the comparison which precedes and that which follows their employment. Moreover as they follow one another very closely in their variations, I would be inclined to recommend that several wires should be compared at the same time that others are used in the field so as to establish the variations that the latter undergo.

It is necessary also to consider that, as the length of the line is calculated from one of the two first equations of (39), we have generally for the difference (A − B) a factor which is greater than unity and consequently it is very advantageous for the result that this difference (A − B) is not influenced by the variations of the wires since they both *vary the same amount.*

The advantages which this method of measuring a base presents are chiefly the following:—

1st. It is evidently very little dependent on the obstacles presented by the ground which may be very hilly without inconvenience. It allows of crossing large ravines, rivers, &c. without difficulty and without costly preparations such as bridges. The base of Moloskovitzi about 10 kilometres long, extends near the village over 4 kilometres of field and plain, then it traverses 3 kilometres of forest where a road has been cut with the axe. Near the north end at Osertizi the base traverses a very undulating space where the mean inclination is well marked and where the depressions between the undulations are of a considerable depth.

2nd. It is much more rapid than other methods: in the work at Moloskovitzi 6 kilometres were once measured in a day. With an experienced establishment and with the work well in train there is nothing to hinder 600 or 800 metres per hour being done, supposing the ground does not offer too great difficulties.

3rd. In a net-work of triangulation we can measure several bases even of considerable length. In general with other forms of apparatus we must be content with bases short in length and few in number, either because the difficulties of the ground do not allow of others being measured or else the expense in time and money is too great. A net-work of triangulation as we know gains a greater uniformity with regard to the partition of errors when the number of bases is increased. On the whole we thus obtain a greater exactitude.

Comparison with the Standard.

This should naturally be made under conditions as like as possible to those under which the work in the field takes place. One would therefore prefer to compare the wires with the standard by repeated measurements of a short base already measured with another apparatus so that, the operation being done at different degrees of temperature, one could determine immediately by these observations the co-efficients of equations (39) or (38) for (A − B) and $\frac{(A-B)^2}{A}$.

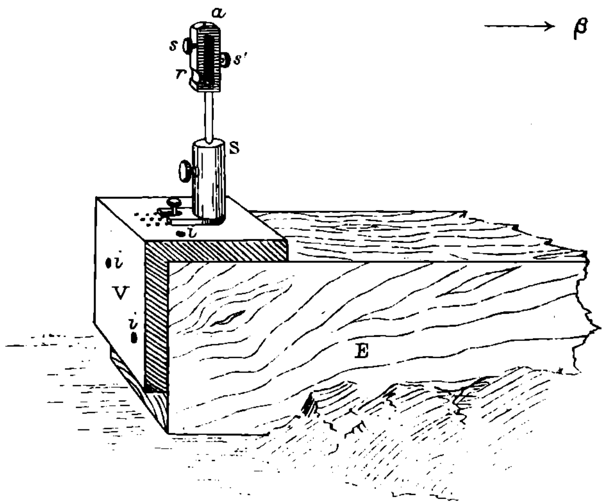

Fig. 4.

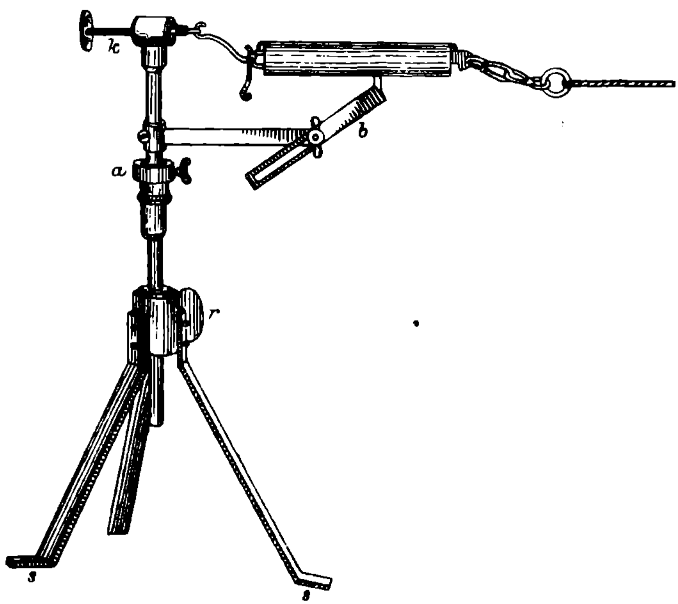

Fig. 5.

It is necessary however to remark that on the one hand such a comparison would be too fatiguing and would need the assistance of too great a number of persons and on the other hand that it is not desirable to be dependent on another measuring apparatus. Such a comparison in fact executed second hand loses in exactitude by the fact that the little errors inevitably made with the apparatus in question exercise an influence on the results obtained with the wires, and consequently the mean measure of the wires has, in this case, no more than a secondary value.

Nevertheless I have mostly been content to compare by measuring the line (page 26) of 100 metres in the garden of the Polytechnic School at Stockholm, although this line had been measured with the apparatus belonging to the Academy of Sciences. For the marking of this line, the changes it underwent, &c., see the memoirs already quoted of *Geodätische Längenmessung* . . . and of *Märklig längdförändring* . . . None of the measurements whether with bars or with wires were made in the sun's rays but either under a cloudy sky or in the afternoon in the shade of high houses. The temperature of the wires was assimilated to that of the surrounding air which was measured with three carefully compared thermometers. These thermometers were placed at the same height from the ground as the wires, one at each end of the base and one in the middle.

Another kind of comparison was undertaken in 1888 conjointly with the measurement of the base of Moloskovitzi. It is described in General Bonsdorff's work which I have already mentioned. The apparatus employed then, the *comparateur*, had been made in Stockholm.

The wires were compared directly with a standard of 25 metres obtained in the following manner in the same riding-school which I have mentioned above. The length of the place was about 70 metres, its breadth 20 metres and its height considerable. The thick walls, the nature of the soil and the great size of the hall acted so that the temperature remained almost constant, the variation not exceeding $0^\circ \cdot 1$ during several hours, whatever number of persons was present, a state of things eminently favourable to comparison. Besides, the temperature was distributed so uniformly that the two extremities of the hall never differed by more than $0^\circ \cdot 2$. On the ground and partly buried was placed a heavy square beam $25^m.3$ long and composed of three pieces put together by means of solid bolts E (fig. 4). At the two ends of this piece of wood thick corners V were fixed composed of two plates of square iron adjusted square with each other so that one of the plates covered the end surface of the beam and the other rested on the upper surface. The corners being strongly fastened to the wood by at least four thumb-screws in the holes i: there is fixed on each of them a sort of shaft S whose nature will be gathered from the drawing. At the top of this shaft which is silvered and slightly rounded there is a cross-mark a which forms one of the extremities of the standard 25 metres. The other extremity is β at the opposite end of the beam on a similar shaft. The apparatus or rather the cross-marks a and β can be adjusted in all directions. In length the two opposing screws s and s' permit of a delicate adjustment. One of the head-pieces is thinned at r so as to make a certain amount of spring when the screws are tightened.

The comparison of the wires with the standard $a\beta$ should certainly be made as in the field in the ordinary way, that is to say manipulating the spring-balance by hand. But for greater accuracy as well as to gain time in comparison, not to mention the economy in assistants, the spring-balance is fastened to a support such as shown in fig. 5. This support is placed beyond the end of the beam E (fig. 4) in a hole made in the ground: it remains there fixed almost immovably thanks to three stakes driven into the ground on which the feet of the support rest. The feet are iron and pierced with holes s through which screws are firmly fixed into the stakes. These supports have means of adjustment in every direction, *viz.*, by means of the ball and socket at a, of the pinching screw at r which allows the shaft to be adjusted to the desired height &c. The spring-balance can be put in tension by means of the screw shaft k, and lastly the disadvantageous friction of the spring-balance is eliminated by the sustaining arm *b*. One person at each end of the standard is sufficient.

It must be remarked that we need not fear a little constant error of comparison caused by the circumstance that the spring-balance is here fitted to the support already mentioned, instead of being in the hand as in field work. For it is shown on page 23 and by equation (40) that the errors of the spring-balance are almost eliminated in the latter case.

In order that the wire, stretched with a force of 10 kilogrammes, may hang free in air and not touch the beam, it is necessary that the shafts S (fig. 4) should be of a height at the least greater than that equivalent to p in equation (16′).

It remains now to determine the absolute length of $\alpha\beta$. In fact this distance may be considered as a miniature base which must be measured with great accuracy.

For this purpose we use a graduated *comparison-standard* of forged Bessemer steel, length $2^{m}.50$ and with a transverse section in the form of an X, the same form as the copies of the metre issued from the International Bureau of weights and measures but greater in thickness. At each half metre of the surface which forms the bottom of the top depression of the X a silver dot is let in. On these dots the graduations are traced. The standard-bar is enclosed in a box which isolates it and shades it from variations of temperature while its graduated extremities project a little from the box. Two attached thermometers determine the temperature.

The bar is placed box and all on two little supports with three feet (figs. 6 and 7) resting on the beam. They ought to support it at two points separated from each other by Bessel's distance. The wooden case is sheathed below with brass plates which are exactly placed at the above-named distance. One of the brass plates is meant to rest on the cylindrical roller in fig. 7 and the other on that in fig. 6 which is provided in the middle with a conical enlargement. These rollers can be adjusted for height and in the lateral direction. Besides, that of fig. 7 has a slow movement of rotation round its own axis by means of the pinion *m* working in the toothed plate *r*. When the roller turns slowly the bar which rests on it is thus pushed slowly in the direction of its length. A turn of the head of the screw *v* only advances the roller along its own axis, that is at right angles to the bar.

This comparison-standard was itself compared in the physical laboratory of the Academy of Sciences of St. Petersburg with the metre standard of the Meteorological Observatory made by Turrittini and examined at Paris. In this operation, as well as in the measurement of the line $\alpha\beta$, the standard was laid in its box supported in all its length and firmly screwed to the bottom at several places.

To align the bar in measuring $\alpha\beta$ two telescopes are used one acting as a collimator for the other. Both have a focal length of about 250 millimetres and an object glass of 60 millimetres. One is furnished with the ordinary diaphragm and the other has a little steel ring fastened on a glass plane in the focus of the object glass, in other words the diaphragm is replaced by an ordinary ring micrometer. The two telescopes were naturally always adjusted to solar focus.

The telescope with the diaphragm was fixed on the box of the standard on one of its outside faces. The catches which held it were movable and adjustable so that it was possible to place the axis of collimation parallel to the line which joined the two extreme crosses of the graduation of the standard. This graduation had been so executed as to make sure that the line in question was parallel to the bar itself which was made rectilinear by planing. The tube of the telescope fixed to the bar is surrounded by two turned rings making part of a cylinder which envelopes the tube in the way usual in levelling telescopes.

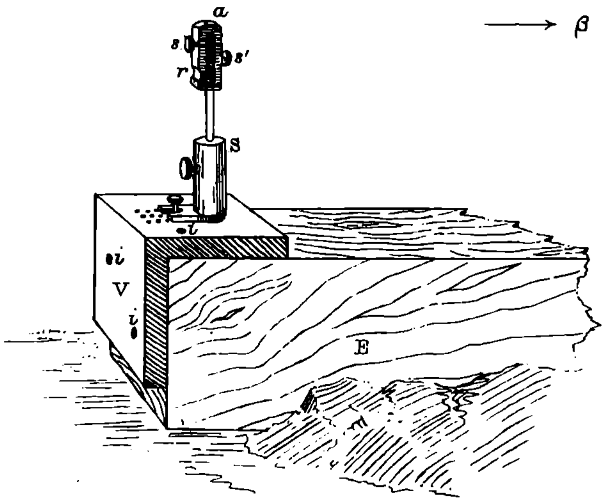

Fig. 4.

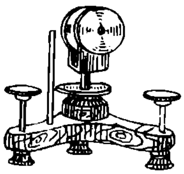

Fig. 6.

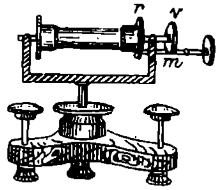

Fig. 7.

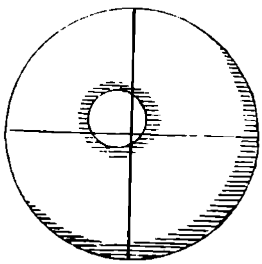

Fig. 8.

To adjust the telescope parallel to the bar it was taken off and placed, resting on its rings, against the inclined planes of the upper groove of the X of the bar. The common axis of the rings was then parallel to the direction of the length of the bar. The telescope on the bar was then pointed to a distant point on the horizon and the diaphragm was centred by turning the telescope round the axis of the rings. The collimation axis was then parallel to the length of the bar and the latter directed to the distant point on the horizon. The telescope was then attached to the exterior of the box and adjusted to the point in the horizon, the bar remaining fixed.

At the other end of the beam, the β side, the collimating telescope was fixed. If then the bar on its rollers was adjusted so that the diaphragm exactly coincided with the centre of the ring micrometer, it would be certain that the bar always advanced in parallel lines.

The measurement with the comparison-standard was executed as follows:—One of the microscopes was adjusted over the mark a of the shaft S (the apparatus admits of three microscopes which can be screwed on the beam at will): then the shaft was removed and one of the terminal marks of the bar placed under the microscope, the bar being placed in the required direction by means of the telescope. Above the other end of the bar another microscope was adjusted: then the bar was taken forward and the operation continued in the ordinary way to the other end of the line. On arrival, that is to say the tenth time, the microscope having been placed above the forward end of the bar, the latter was taken away and the mark β of the shaft adjusted under the microscope.

In this case the line $a\beta$ was not determined beforehand, but only the point a. If β had also been fixed before the measurement, it would have been necessary on the arrival of the bar to place first the microscope above the point β, then take away the shaft S, then adjust the bar under the microscope without using the telescope, then replace the shaft with the mark β in the place it occupied before, the bar being naturally placed on one side. It is probable that then the diaphragm of the bar telescope would not coincide with the centre of the collimating ring. Let us suppose that the internal angular radius of this ring is r. If it happens that the diaphragm meets the internal edge of the ring, the bar will make an angle r with the true direction and the length $2^{m}.50$ of this bar being called l, we get a correction equivalent to

$$-l(1-\cos r) = -\frac{l}{2}\cdot r^2 + \&c.$$

If again the diaphragm does not cut the edge of the ring but a point at a distance nr from the centre of the ring (fig. 8), the correction to the length will be

$$m = -\frac{r^2 l}{2} n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where $\frac{r^2 l}{2}$ is constant and fixed once for all; n can be estimated by eye with sufficient accuracy as the following example shows. Suppose that $l = 2^{m}.50$, $r = 6' \, 12'' = \cdot 00180$, then the constant will be $0^{mm}.00407$. If moreover n as in fig. 8 is estimated at $0\cdot 6$ we have

$$m = -0^{mm}.00407 \times 0\cdot 36 = -0^{mm}.001$$

or a quantity determined with more than sufficient accuracy.

The correction arising from the divergence from the line $a\beta$ of the nine first lengths of the bar $\left(\text{angle} = \frac{nr}{9}\right)$ will be nine times smaller than that which we have just determined, that is to say in every case absolutely imperceptible.

It must be remarked that it is not necessary that the telescope of the comparison bar should be exactly parallel to the bar itself.

To try the resistance of the beam to an accidental pressure more or less intense, the following experiment was made at St. Petersburg during the above-mentioned comparison. The line $\alpha\beta$ was first compared with four wires. Then two persons placed themselves on the beam making violent movements. At the same time and afterwards the wires were again compared with $\alpha\beta$ and no difference in the readings could be noted. So that we may be convinced that the operations with the comparison bar do not disarrange the beam in an appreciable fashion.

In most cases the wires were first compared with the line $\alpha\beta$: then this having been measured with the standard, the wires were again compared. The temperature of the wires being supposed the same as the surrounding air and read by thermometers, can be determined more exactly in a closed place than in the open air.

The probable error of a single complete comparison of wires never exceeds $0^{mm}.04$.

The wires 50 metres in length cannot of course be compared in the same way: but seeing that they are used only exceptionally and only to traverse wide difficult pieces of ground, it does not matter much if their comparison is not executed with rigorous accuracy. The simplest method is the following which was applied at St. Petersburg. Three tripods were placed in the riding-school as in fig. 1 in a straight line and at mutual distances of 25 metres. The two intervals of 25 metres thus obtained were measured in the ordinary way with wires of this length, already compared; at the same time the supports were levelled. It was then possible to calculate the rectilinear distance between the extreme supports. The middle support was then taken away and the distance between the two extreme ones measured with the 50 metre wires.

As the geodetic standards issued by the International Bureau of weights and measures are 4 metres long, the comparison standard bar should perhaps also be this length. In that case the length of the wires ought to be calculated by a multiple of 4 metres and should be, for example, 24 instead of 25 metres.

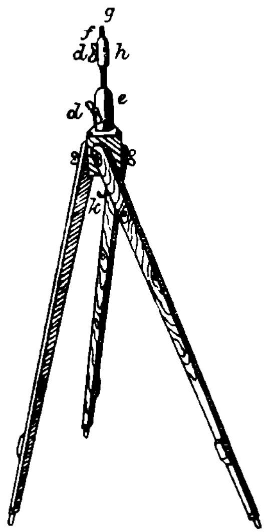

Fig. 1.